Mathematics 160: Lecture 24

The Cross Product

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Definition

- Recall: if
$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix},$$
then
$$\vec{v} \times \vec{w} = \det \begin{bmatrix} \vec{\imath} & x_1 & x_2 \\ \vec{\jmath} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{bmatrix} = \begin{bmatrix} y_1 z_2 - y_2 z_1 \\ x_2 z_1 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}.$$

Some properties

- It follows immediately from the definition that, for any vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and any scalar a,
  - $\vec{v} \times \vec{0} = \vec{0}$,
  - $\vec{v} \times \vec{v} = \vec{0}$,
  - $\vec{v} \times \vec{w} = -(\vec{w} \times \vec{v})$,
  - $(a\vec{v}) \times \vec{w} = a(\vec{v} \times \vec{w}) = \vec{v} \times (a\vec{w})$.
- Also, for any third vector $\vec{u}$, if A is the matrix with columns $\vec{u}$, $\vec{v}$, and $\vec{w}$, then
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det A.$$
- We call the latter the *scalar triple product* of $\vec{u}$, $\vec{v}$, and $\vec{w}$.
- Homework exercise:
  - $\vec{v} \times (\vec{u} + \vec{w}) = (\vec{v} \times \vec{u}) + (\vec{v} \times \vec{w})$
  - $(\vec{u} + \vec{w}) \times \vec{v} = (\vec{u} \times \vec{v}) + (\vec{w} \times \vec{v})$

The Lagrange identity

- Suppose
$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}.$$
- Then
$$\begin{aligned} \|\vec{v} \times \vec{w}\|^2 &= (y_1 z_2 - y_2 z_1)^2 + (x_2 z_1 - x_1 z_2)^2 + (x_1 y_2 - x_2 y_1)^2 \\ &= y_1^2 z_2^2 - 2y_1 y_2 z_1 z_2 + y_2^2 z_1^2 + x_2^2 z_1^2 - 2x_1 x_2 z_1 z_2 + x_1^2 z_2^2 \\ &\quad + x_1^2 y_2^2 - 2x_1 x_2 y_1 y_2 + x_2^2 y_1^2 \\ &= (x_1^2 + y_1^2 + z_1^2)(x_2^2 + y_2^2 + z_2^2) - (x_1^2 x_2^2 + y_1^2 y_2^2 + z_1^2 z_2^2) \\ &\quad - 2(x_1 x_2 y_1 y_2 + x_1 x_2 z_1 z_2 + y_1 y_2 z_1 z_2) \\ &= \|\vec{v}\|^2 \|\vec{w}\|^2 - (x_1 x_2 + y_1 y_2 + z_1 z_2)^2 \\ &= \|\vec{v}\|^2 \|\vec{w}\|^2 - (\vec{v} \cdot \vec{w})^2. \end{aligned}$$

Geometry

- It now follows that if θ is the angle between $\vec{v}$ and $\vec{w}$, then

$$\begin{aligned}\|\vec{v}\times\vec{w}\|^2 &= \|\vec{v}\|^2\|\vec{w}\|^2 - (\|\vec{v}\|\|\vec{w}\|\cos(\theta))^2 \\ &= \|\vec{v}\|^2\|\vec{w}\|^2(1-\cos^2(\theta)) \\ &= \|\vec{v}\|^2\|\vec{w}\|^2\sin^2(\theta).\end{aligned}$$

- Since $0 \le \theta \le \pi$, $\sin(\theta) \ge 0$, so we have

$$\|\vec{v}\times\vec{w}\| = \|\vec{v}\|\|\vec{w}\|\sin(\theta).$$

- Note: if P is the parallelogram with adjacent sides $\vec{v}$ and $\vec{w}$, then $\|\vec{w}\|\sin(\theta)$ is the height of P. Hence

$$\|\vec{v}\times\vec{w}\| = \text{ area of } P.$$

Example

- Let A be the area of the triangle with vertices at

$$\vec{p} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{q} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}, \text{ and } \vec{r} = \begin{bmatrix} 3 \\ 1 \\ 5 \end{bmatrix}.$$

- Let

$$\vec{v} = \vec{q} - \vec{p} = \begin{bmatrix} -3 \\ 2 \\ 4 \end{bmatrix} \text{ and } \vec{w} = \vec{r} - \vec{p} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}.$$

- Then

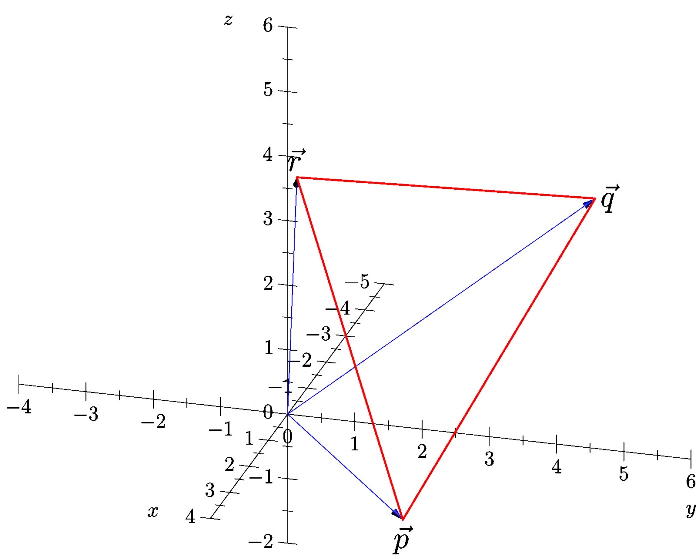
$$\vec{v}\times\vec{w} = \det\begin{bmatrix} \vec{\imath} & -3 & 2 \\ \vec{\jmath} & 2 & -1 \\ \vec{k} & 4 & 6 \end{bmatrix} = \begin{bmatrix} 16 \\ 26 \\ -1 \end{bmatrix}$$

and

$$A = \frac{1}{2}\|\vec{v}\times\vec{w}\| = \frac{1}{2}\sqrt{933}.$$

Example (cont'd)

- Triangle from above:



Right-hand rule

- Homework exercise: $\vec{\imath}\times\vec{\jmath} = \vec{k}$, $\vec{\jmath}\times\vec{k} = \vec{\imath}$, and $\vec{k}\times\vec{i} = \vec{j}$.
- In general, $\vec{v}\times\vec{w}$ is a vector of length $\|\vec{v}\|\|\vec{w}\|\sin(\theta)$, orthogonal to both $\vec{v}$ and $\vec{w}$, in the direction determined by the *right-hand rule*.

Areas and determinants

- Let P be a parallelogram in $\mathbb{R}^2$ with adjacent sides
$$\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$
- Let
$$\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix} \text{ and } \vec{w_1} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}.$$
- If A is the area of P, then
$$A = \|\vec{v_1} \times \vec{w_1}\| = |x_1 y_2 - x_2 y_1| = \left| \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \right|.$$

Example

- If A is the area of a parallelogram in $\mathbb{R}^2$ with adjacent sides
$$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$
then
$$A = \left| \det \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \right| = 5.$$

Parallelpipeds

- Let P be the parallelepiped with adjacent sides $\vec{u}$, $\vec{v}$, and $\vec{w}$.
- With the parallelogram having adjacent sides $\vec{v}$ and $\vec{w}$ as base, the height of P is the absolute value of the component of $\vec{u}$ in the direction of $\vec{v} \times \vec{w}$:
$$\frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|}.$$
- Hence the volume V of P is
$$V = \frac{|\vec{u} \cdot (\vec{v} \times \vec{w})|}{\|\vec{v} \times \vec{w}\|} \|\vec{v} \times \vec{w}\| = |\vec{u} \cdot (\vec{v} \times \vec{w})|.$$
- That is, the volume of P is the absolute value of the scalar triple product of $\vec{u}$, $\vec{v}$, and $\vec{w}$, or, equivalently, the absolute value of he determinant of the matrix with $\vec{u}$, $\vec{v}$, and $\vec{w}$ for columns.

Example

- Let P be the parallelepiped with adjacent sides
$$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}.$$
- Since
$$\det \begin{bmatrix} 0 & 1 & -3 \\ 1 & 4 & 1 \\ 5 & 1 & 1 \end{bmatrix} = -(1+3) + 5(1+12) = 61,$$
the volume of P is 61.

Example (cont'd)

- Parallelepiped from above: