Mathematics 160: Lecture 25

Subspaces

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Lines through the origin

- Recall: if ℓ is a line through the origin in $\mathbb{R}^n$ in the direction of $\vec{d}$, then

$$\ell = \{t\vec{d} : -\infty < t < \infty\}.$$

- Note: if $\vec{v}$ and $\vec{w}$ are on ℓ, then $\vec{v} = s\vec{d}$ and $\vec{w} = t\vec{d}$ for some scalars s and t, and so $\vec{v} + \vec{w}$ is on ℓ since

$$\vec{v} + \vec{w} = s\vec{d} + t\vec{d} = (s+t)\vec{d}.$$

- Similarly, for any scalar a, $a\vec{v}$ is on ℓ since

$$a\vec{v} = (as)\vec{d}.$$

Planes through the origin

- Recall: if $\mathcal{P}$ is a plane through the origin in $\mathbb{R}^n$, then, for some vectors $\vec{v}$ and $\vec{w}$,

$$\mathcal{P} = \{s\vec{v} + t\vec{w} : -\infty < s < \infty, -\infty < t < \infty\}.$$

- If $\vec{x}$ and $\vec{y}$ are both on $\mathcal{P}$, then, for some scalars s_1, t_1, s_2, and t_2,

$$\vec{x} = s_1\vec{v} + t_1\vec{w} \text{ and } \vec{y} = s_2\vec{v} + t_2\vec{w}.$$

- Hence $\vec{x} + \vec{y}$ is on $\mathcal{P}$ since

$$\vec{x} + \vec{y} = (s_1 + s_2)\vec{v} + (t_1 + t_2)\vec{w}.$$

- Similarly, for any scalar a, $a\vec{x}$ is on $\mathcal{P}$ since

$$a\vec{x} = (as_1)\vec{v} + (at_1)\vec{w}.$$

Definition

- We say a subset U of $\mathbb{R}^n$ is a *subspace* of $\mathbb{R}^n$ if
  - the zero vector $\vec{0}$ is in U,
  - whenever $\vec{v}$ and $\vec{w}$ are in U, $\vec{v} + \vec{w}$ is also in U, and
  - whenever $\vec{v}$ is in U and a is a scalar, $a\vec{v}$ is also in U.
- Note: $U = \{\vec{0}\}$ and $U = \mathbb{R}^n$ are both subspaces of $\mathbb{R}^n$. We call any other subspace of $\mathbb{R}^n$ a *proper* subspace.

Examples

- We have already shown that every line through the origin and every plane through the origin is a subspace.
- A line or plane which does not pass through the origin is not a subspace.
- The set

$$U = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \right\}$$

is not a subspace of $\mathbb{R}^2$.

Null spaces

- If A is an $m \times n$ matrix, we call

$$\text{null}\, A = \{\vec{x} : \vec{x} \in \mathbb{R}^n, A\vec{x} = \vec{0}\}.$$

the *null space* of A
- That is, $\text{null}\, A$ is the set of all solutions of the homogeneous linear system $A\vec{x} = \vec{0}$.
- If $\vec{x}$ and $\vec{y}$ are in $\text{null}\, A$, and a is a scalar, then

$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$$

and

$$A(a\vec{x}) = aA\vec{x} = a\vec{0} = \vec{0}.$$

- Also, $A\vec{0} = \vec{0}$, so $\vec{0}$ is in $\text{null}\, A$.
- Thus $\text{null}\, A$ is a subspace of $\mathbb{R}^n$.

Image spaces

- If A is an $m \times n$ matrix, we call

$$\text{im}\, A = \{\vec{y} : \vec{y} \in \mathbb{R}^m, \vec{y} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}.$$

the *image space* of A.
- That is, $\text{im}\, A$ is the set of all vectors $\vec{y}$ in $\mathbb{R}^m$ for which the linear system $A\vec{x} = \vec{y}$ has a solution.

Image spaces (cont'd)

- $\vec{0}$ is in $\text{im}\, A$ since $A\vec{0} = \vec{0}$.
- If $\vec{y}_1$ and $\vec{y}_2$ are in $\text{im}\, A$, then there exist $\vec{x}_1$ and $\vec{x}_2$ in $\mathbb{R}^n$ such that $A\vec{x}_1 = y_1$ and $A\vec{x}_2 = y_2$. Hence

$$A(\vec{x}_1 + \vec{x}_2) = A\vec{x}_1 + A\vec{x}_2 = \vec{y}_1 + \vec{y}_2,$$

so $\vec{y}_1 + \vec{y}_2$ is in $\text{im}\, A$.
- Moreover, if a is a scalar, then $A(a\vec{x}_1) = aA\vec{x}_1 = a\vec{y}_1$, and so $a\vec{y}_1$ is in $\text{im}\, A$.
- Thus $\text{im}\, A$ is a subspace of $\mathbb{R}^m$.

Eigenspaces

- Suppose A is an $n \times n$ matrix.
- For a scalar λ, let
$$E_\lambda(A) = \{\vec{x} : A\vec{x} = \lambda\vec{x}\}.$$
- Since $E_\lambda(A) = \text{null}(\lambda I - A)$, $E_\lambda(A)$ is a subspace of $\mathbb{R}^n$, which we call the *eigenspace* of A corresponding to λ.
- Note: $E_\lambda(A) = \{\vec{0}\}$ unless λ is an eigenvalue of A.

Definition

- Given k vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$ in $\mathbb{R}^n$, we call the set
$$\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} = \{t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k : t_1, t_2, \ldots, t_k \in \mathbb{R}\}$$
the *span* of $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$.
- That is, the span of a set of vectors is the set of all linear combinations of the vectors.

Examples

- If $\vec{x}_1$ is a nonzero vector in $\mathbb{R}^n$, then $\text{span}\{\vec{x}_1\}$ is the line through the origin in the direction of $\vec{x}_1$.
- If $\vec{x}_1$ and $\vec{x}_2$ are two vectors in $\mathbb{R}^n$, then $\text{span}\{\vec{x}_1, \vec{x}_2\}$ is a plane through the origin if $\vec{x}_1$ and $\vec{x}_2$ are not parallel and a line through the origin, otherwise.
- If $\vec{e}_1$, $\vec{e}_2$, ..., $\vec{e}_n$ are the coordinate vectors in $\mathbb{R}^n$, then
$$\text{span}\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\} = \mathbb{R}^n.$$

Example

- Let
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$
- Let $U = \text{span}\{\vec{x}_1, \vec{x}_2\}$.
- Then, for example, $\vec{v} = \begin{bmatrix} -2 & 4 \end{bmatrix}^T$ is in U if and only if $\vec{v} = s\vec{x}_1 + t\vec{x}_2$ for some scalars s and t.
- That is, $\vec{v}$ is in U if and only if we can solve
$$\begin{aligned} s + 3t &= -2 \\ s + t &= 4. \end{aligned}$$
- Solving, we find $s = 7$ and $t = -3$, so $\vec{v}$ is in $\text{span}\{\vec{x}_1, \vec{x}_2\}$.

Example (cont'd)

- Note: $\vec{w} = \begin{bmatrix} a & b \end{bmatrix}^T$ is in span$\{\vec{x}_1, \vec{x}_2\}$ as long as we can solve

$$A \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}, \text{ where } A = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}.$$

- Since A is invertible, the equation has a solution for any choice of $\vec{w}$.
- Hence span$\{\vec{x}_1, \vec{x}_2\} = \mathbb{R}^2$.

Theorem

- Given k vectors $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$ in $\mathbb{R}^n$,

$$U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$$

  is a subspace of $\mathbb{R}^n$.

Theorem (cont'd)

- Reason:
  - $\vec{0}$ is in U since $\vec{0} = 0\vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_k$.
  - Suppose $\vec{v}$ and $\vec{w}$ are in U and a is a scalar. Then

$$\vec{v} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$

    for some scalars t_1, t_2, ..., t_k, and

$$\vec{w} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k$$

    for some scalars s_1, s_2, ..., s_k.
  - Then

$$\vec{v} + \vec{w} = (t_1 + s_1)\vec{x}_1 + (t_2 + s_2)\vec{x}_2 + \cdots + (t_k + s_k)\vec{x}_k$$

    and

$$a\vec{v} = at_1\vec{x}_1 + at_2\vec{x}_2 + \cdots + at_k\vec{x}_k,$$

    so $\vec{v} + \vec{w}$ and $a\vec{v}$ are in U.
  - Hence U is a subspace of $\mathbb{R}^n$.

Theorem

- If $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$ are in a subspace U of $\mathbb{R}^n$, then

$$\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} \subseteq U.$$

Null spaces and image spaces

- Null spaces:
  - Suppose A is an $m \times n$ matrix and $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$ are basic solutions to the homogeneous equation $AX = O$.
  - Then
$$\text{null}\, A = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots \vec{x}_k\}.$$

- Image spaces:
  - Suppose A is an $m \times n$ matrix and $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_n$ are the columns of A.
  - Then
$$\text{im}\, A = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}.$$