Mathematics 160: Lecture 26

Linear Independence

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Example

- Let
$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \text{ and } \vec{x}_3 = \begin{bmatrix} -5 \\ 8 \\ 3 \\ 14 \end{bmatrix}.$$
- Note: $\vec{x}_3 = 2\vec{x}_1 + 3\vec{x}_2$.
- Hence $\text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \text{span}\{\vec{x}_1, \vec{x}_2\}$.
- Note: $\vec{x}_2$ is not a scalar multiple of $\vec{x}_1$.
- We say $\vec{x}_1$ and $\vec{x}_2$ are *linearly independent*, whereas $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are *linearly dependent*.

Example

- Let
$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix} -3 \\ 2 \\ -1 \\ 4 \end{bmatrix},$$
as in the previous example.
- Let $\mathcal{P}$ be the plane with vector equation $\vec{p} = s\vec{x}_1 + t\vec{x}_2$.
- Suppose $\vec{q}$ is a point in $\mathcal{P}$. Then $\vec{q} = s_1\vec{x}_1 + t_1\vec{x}_2$ for some scalars s_1 and t_1.
- Suppose we could also write $\vec{q} = s_2\vec{x}_1 + t_2\vec{x}_2$, where s_2 and t_2 are again scalars.
- We would then have
$$s_1\vec{x}_1 + t_1\vec{x}_2 = s_2\vec{x}_1 + t_2\vec{x}_2,$$
implying that
$$(s_1 - s_2)\vec{x}_1 + (t_1 - t_2)\vec{x}_2 = \vec{0}.$$

Example (cont'd)

- Unless both $s_1 - s_2 = 0$ and $t_1 - t_2 = 0$, this would imply that $\vec{x}_1$ and $\vec{x}_2$ are parallel.
- Since $\vec{x}_1$ and $\vec{x}_2$ are not parallel, we must have $s_1 = s_2$ and $t_1 = t_2$.
- That is, every point $\vec{p}$ in the plane $\mathcal{P}$ may be written in one and only one way as a linear combination of $\vec{x}_1$ and $\vec{x}_2$.

Definition

- We say a set of vectors $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ in $\mathbb{R}^n$ is *linearly independent* if

$$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0},$$

for some scalars t_1, t_2, ..., t_k, implies

$$t_1 = t_2 = \cdots = t_k = 0.$$

Otherwise, we say $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is *linearly dependent*.

Example

- Consider the vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \text{ and } \vec{x}_3 \begin{bmatrix} 4 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

in $\mathbb{R}^4$.
- To test $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ for linear independence, we look for scalars r, s, and t for which

$$r\vec{x}_1 + s\vec{x}_2 + t\vec{x}_3 = \vec{0}.$$

Example (cont'd)

- That is, we are asking if the system

$$\begin{aligned} r - 2s + 4t &= 0 \\ 3r - 2t &= 0 \\ r + s - t &= 0 \\ 2r - s &= 0 \end{aligned}$$

has only the trivial solution.
- We have

$$\begin{bmatrix} 1 & -2 & 4 & 0 \\ 3 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 \\ 2 & -1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 4 & 0 \\ 0 & 1 & -\frac{7}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Hence the only solution is the trivial solution $r = s = t = 0$, and so $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ is an independent set of vectors in $\mathbb{R}^4$.

Theorem

- If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then for every $\vec{y}$ in span$\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ there exist unique scalars t_1, t_2, ..., t_k such that

$$\vec{y} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k.$$

Theorem (cont'd)

- Reason:
  - Suppose we have both
$$\vec{y} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k$$
    and
$$\vec{y} = s_1\vec{x}_1 + s_2\vec{x}_2 + \cdots + s_k\vec{x}_k.$$
  - Subtracting, we have
$$(t_1 - s_1)\vec{x}_1 + (t_2 - s_2)\vec{x}_2 + \cdots + (t_k - s_k)\vec{x}_k = \vec{0}.$$
  - It follows from the linear independence of $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ that
$$t_1 - s_1 = 0, t_2 - s_2 = 0, \ldots, t_k - s_k = 0.$$

Theorem

- Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$. If U is an $n \times n$ invertible matrix, then $\{U\vec{x}_1, U\vec{x}_2, \ldots, U\vec{x}_k\}$ is also a linearly independent set of vectors.

Theorem (cont'd)

- Reason:
  - Suppose that, for some scalars t_1, t_2, ..., t_k,
$$t_1 U\vec{x}_1 + t_2 U\vec{x}_2 + \cdots + t_k U\vec{x}_k = \vec{0}.$$
  - Then
$$\vec{0} = U(t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k),$$
    and so, since U is invertible,
$$\vec{0} = t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k.$$
  - Since $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent, it follows that
$$t_1 = t_2 = \cdots = t_k = 0.$$
  - Hence $\{U\vec{x}_1, U\vec{x}_2, \ldots, U\vec{x}_k\}$ is linearly independent.

Theorem

- Suppose R is an $m \times n$ matrix in row-echelon form and let $\vec{y}_1$, $\vec{y}_2$, ..., $\vec{y}_k$ be the nonzero rows of R. Then $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$.

Theorem (cont'd

- Reason:
  - Suppose, for some scalars t_1, t_2, ..., t_k,
    $$t_1\vec{y}_1 + t_2\vec{y}_2 + \cdots + t_k\vec{y}_k = \vec{0}.$$
  - Suppose the first leading 1 in R is in column j. Then the jth entry of $\vec{y}_1$ is 1 and the jth entry of $\vec{y}_i$, $i = 2, 3, \ldots, k$, is 0.
  - Hence we must have $t_1 = 0$.
  - We then have
    $$t_2\vec{y}_2 + \cdots + t_k\vec{y}_k = \vec{0}.$$
  - Proceeding as above, we show, successively, that $t_2 = 0$, $t_3 = 0$, and so on, up to $t_k = 0$.
  - Hence $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ is linearly independent.

Theorem

- Suppose A is an $n \times n$ matrix. The following are equivalent:
  - A is invertible.
  - The columns of A are linearly independent in $\mathbb{R}^n$.
  - The columns of A span $\mathbb{R}^n$.
  - The rows of A are linearly independent in $\mathbb{R}^n$.
  - The rows of A span $\mathbb{R}^n$.