Mathematics 160: Lecture 27

Bases

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Theorem

- Let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ be a subspace of $\mathbb{R}^n$ and suppose $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ is a linearly independent set of vectors in U. Then $k \leq m$.

Theorem (cont'd)

- Reason:
  - Since, for $j = 1, 2, \ldots, k$, $\vec{y}_j$ is in U, there exist scalars a_{ij}, $i = 1, 2, \ldots, m$, such that

$$\vec{y}_j = a_{1j}\vec{x}_1 + a_{2j}\vec{x}_2 + \cdots + a_{mj}\vec{x}_m.$$

  - Let $A = [a_{ij}]$, an $m \times k$ matrix.
  - Suppose $k > m$. Then the system $A\vec{t} = \vec{0}$ has a nontrivial solution

$$\vec{t} = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix}.$$

Theorem (cont'd)

- Reason (cont'd):
  - Let B be the $n \times k$ matrix with columns $\vec{y}_j$, $j = 1, 2, \ldots, k$, and let C be the $n \times m$ matrix with columns $\vec{x}_i$, $i = 1, 2, \ldots, m$.
  - Then $B = CA$, and so

$$t_1\vec{y}_1 + t_2\vec{y}_2 + \cdots + t_k\vec{y}_k = B\vec{t} = CA\vec{t} = X\vec{0} = \vec{0},$$

    contradicting the assumption that $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_k\}$ is linearly independent.
  - Hence we must have $k \leq m$.

Some consequences

- If $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.
- If $\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\} = \mathbb{R}^n$, then $k \geq n$.

Definition

- Suppose U is a subspace of $\mathbb{R}^n$. We call a set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ a *basis* of U if
    - $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent, and
    - $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$.

Theorem

- If both $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ and $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m\}$ are bases of a subspace U of $\mathbb{R}^n$, then $k = m$.
- Reason:
    - Since $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent and is in $\text{span}\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m\}$, we have $k \leq m$.
    - Since $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_m\}$ is linearly independent and is in $\text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$, we have $m \leq k$.
    - Hence $k = m$.