Mathematics 160: Lecture 28

Dimension

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Definition

- If U is a subspace of $\mathbb{R}^n$, we say U has *dimension* k if there exists a basis for U with k vectors. We write $\dim U = k$.
- Example
  - We call the set of coordinate vectors $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ the *standard basis* of $\mathbb{R}^n$.
  - In particular, $\dim \mathbb{R}^n = n$.

Theorem

- Suppose $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a linearly independent set of vectors and let $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$. If $\vec{y}$ is not in U, then $\{\vec{y}, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.
- Reason:
  - Suppose, for some scalars s, t_1, t_2, ..., t_k,
  $$s\vec{y} + t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0}.$$
  - Then $s = 0$, since otherwise $\vec{y}$ is a linear combination of $\vec{x}_1$, $\vec{x}_2$, ..., $\vec{x}_k$, contrary to the assumption that $\vec{y}$ is not in U.
  - But then
  $$t_1\vec{x}_1 + t_2\vec{x}_2 + \cdots + t_k\vec{x}_k = \vec{0},$$
  implying that $t_1 = t_2 = \cdots = t_k = 0$.
  - Thus $\{\vec{y}, \vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is linearly independent.

Consequences

- Suppose U is a proper subspace of $\mathbb{R}^n$. Then:
  - Any linearly independent subset of U can be enlarged to a basis of U.
  - U has a basis.
- Note: if $U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$, then some subset of $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ is a basis for U.

Theorem

- Let U and V be subspaces of $\mathbb{R}^n$. Then
  - If $U \subseteq V$, then $\dim U \leq \dim V$.
  - If $U \subseteq V$ and $\dim U = \dim V$, then $U = V$.
  - If $\dim U = d$, then any set of d linearly independent vectors in U is a basis of U.
  - If $\dim U = d$, then any spanning set of U which contains d vectors is a basis of U.

Example

- Since
$$\det\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ -1 & 2 & -1 \end{bmatrix} = -1,$$
the vectors
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \text{ and } \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$
form a basis for $\mathbb{R}^3$.

Example

- Let
$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}.$$
- Let $U = \text{span}\{\vec{v}, \vec{w}\}$.
- So $\dim U = 2$.
- If
$$\vec{x} = \vec{v} - \vec{w} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} \text{ and } \vec{y} = 2\vec{v} + 3\vec{w} = \begin{bmatrix} 8 \\ -4 \\ 11 \end{bmatrix},$$
then $\vec{x}$ and $\vec{y}$ lie in U
- Since $\{\vec{x}, \vec{y}\}$ is linearly independent, it follows that $\{\vec{x}, \vec{y}\}$ is a basis for U.

Theorem

- If U is a subspace of $\mathbb{R}^n$, then
$$U = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$$
for some set $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ of linearly independent vectors.
- In particular, every one-dimension subspace of $\mathbb{R}^n$ is a line through the origin and every two-dimensional subspace of $\mathbb{R}^n$ is a plane through the origin.

Example

- Let $U = \text{im}\, A$ and $V = \text{null}\, A$, where

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 3 & 2 & -1 & 2 \end{bmatrix}$$

- Row reducing the augmented matrix for solving the linear system $A\vec{x} = \vec{0}$, we have

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 2 & 1 & 4 & 0 & 0 \\ 3 & 2 & -1 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 6 & 0 \\ 0 & 0 & 1 & \frac{1}{7} & 0 \end{bmatrix}.$$

- It follows that solutions of $A\vec{x} = \vec{0}$ are of the form

$$\vec{x} = t \begin{bmatrix} \frac{23}{7} \\ -6 \\ -\frac{1}{7} \\ 1 \end{bmatrix}.$$

Example (cont'd)

- Hence

$$\left\{ \begin{bmatrix} 23 \\ -42 \\ -1 \\ 7 \end{bmatrix} \right\}$$

is a basis for V.
- In particular, $\dim V = 1$.

Example (cont')

- Note: U is a subspace of $\mathbb{R}^3$, and so $\dim U \leq 3$.
- Now

$$\det \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 4 \\ 3 & 2 & -1 \end{bmatrix} = 7.$$

- Hence

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\}$$

is a basis for U.
- Hence $\dim U = 3$.
- Note: in fact, $U = \mathbb{R}^3$.