Mathematics 160: Lecture 29

Rank

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Definition

- Let A be an $m \times n$ matrix with columns $\vec{c}_1$, $\vec{c}_2$, ..., $\vec{c}_n$ and rows $\vec{r}_1^T$, $\vec{r}_2^T$, ..., $\vec{r}_m^T$. We call

$$\text{col}\, A = \text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$$

the *column space* of A and

$$\text{row}\, A = \text{span}\{\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_m\}$$

the *row space* of A.
- Note: $\text{col}\, A$ is a subspace of $\mathbb{R}^m$ and $\text{row}\, A$ is a subspace of $\mathbb{R}^n$.
- Note: $\text{col}\, A$ is another name for $\text{im}\, A$.

Theorem

- If A is obtained from B by a sequence of elementary row operations, then $\text{row}\, A = \text{row}\, B$.
- Reason:
  - If B is obtained from A by a single elementary row operation, then the rows of B are linear combinations of the rows of A, and so $\text{row}\, B \subseteq \text{row}\, A$.
  - But then A is obtained from B by a single elementary row operation (the inverse of the operation used to produce B), and so $\text{row}\, A \subseteq \text{row}\, B$.
  - Thus $\text{row}\, A = \text{row}\, B$.

Theorem

- If B is obtained from A by a sequence of elementary column operations, then $\text{col}\, A = \text{col}\, B$.

Theorem

- Suppose A is an $m \times n$ matrix, R is obtained from A by a sequence of elementary row operations, and R is in row-echelon form. Then the nonzero rows of R are a basis for row A.
- As a consequence, rank $A = \dim(\text{row } A)$.
- Note: this verifies our claim in Section 1.2 that the rank of a matrix is well-defined.

Example

- Let
$$\vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \\ 6 \end{bmatrix}, \text{ and } \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 3 \\ -4 \end{bmatrix}.$$
- To find a basis for $U = \text{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$, we note that
$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 3 & 4 & 1 & 6 \\ -1 & 0 & 3 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example (cont'd)

- Hence $\dim U = 2$, and
$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{5}{2} \\ -\frac{3}{2} \end{bmatrix} \right\}$$
is a basis for U.

Column spaces

- Note: if R is an $m \times n$ matrix in row-echelon form, then the columns of R containing leading 1's form a basis for col R.
- In particular, $\dim(\text{row } R) = \dim(\text{col } R)$.
- Now suppose A is an $m \times n$ matrix and R, a row-echelon matrix, is obtained from A by elementary row operations. Then $R = UA$ for some invertible matrix U.
- Let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n$ be the columns of A. Then $U\vec{c}_1, U\vec{c}_2, \ldots, U\vec{c}_n$ are the columns of R.
- Let $U\vec{c}_{j_1}, U\vec{c}_{j_2}, \ldots, U\vec{c}_{j_r}$ be the columns of R which contain leading 1's. From above, $\{U\vec{c}_{j_1}, U\vec{c}_{j_2}, \ldots, U\vec{c}_{j_r}\}$ is a basis for col R.
- It follows, since U is invertible, that $\{\vec{c}_{j_1}, \vec{c}_{j_2}, \ldots, \vec{c}_{j_r}\}$ is linearly independent.

Column spaces (cont')

- Now suppose $\vec{x}$ is in col A. Then

$$\vec{x} = s_1\vec{c}_1 + s_2\vec{c}_2 + \cdots + s_n\vec{c}_n$$

for some scalars s_1, s_2, ..., s_n.

- Then

$$\begin{aligned} U\vec{x} &= s_1 U\vec{c}_1 + s_2 U\vec{c}_2 + \cdots + s_n U\vec{c}_n \\ &= t_1 U\vec{c}_{j_1} + t_2 U\vec{c}_{j_2} + \cdots + t_r U\vec{c}_{j_r} \end{aligned}$$

for some scalars t_1, t_2, ..., t_r since $\{U\vec{c}_{j_1}, U\vec{c}_{j_2}, \ldots, U\vec{c}_{j_r}\}$ is a basis for col R.

- Multiplying both sides by U^{-1},

$$\vec{x} = t_1\vec{c}_{j_1} + t_2\vec{c}_{j_2} + \cdots + t_r\vec{c}_r.$$

- Hence $\{\vec{c}_{j_1}, \vec{c}_{j_2}, \ldots, \vec{c}_{j_r}\}$ spans col A, and so is a basis for col A.

Rank Theorem

- If A is an $m \times n$ matrix, then $\dim(\text{row } A) = \dim(\text{col } A) = \text{rank } A$.

Example

- Let

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}.$$

- Using elementary row operations,

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example (cont'd)

- Hence

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

is a basis for row A and

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

is a basis for col A.