# Mathematics 160: Lecture 3

## Gaussian Elimination

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## Coefficient matrix

- The *coefficient matrix* of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is the $m \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$



## Augmented matrix

- The *augmented matrix* of the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

is the $m \times (n+1)$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}.$$



## Row operations

- Idea: the operations we used to solve a system of linear equations affected only the coefficients and the constants. Hence we can solve the system more easily by working directly with the augmented matrix.
- The *elementary row operations* are
  - interchange rows,
  - multiply a row by nonzero scalar,
  - add a multiple of one row to another row.
- Important: Applying a row operation produces an equivalent system (that is, a system with exactly the same solution set).

## Example

- Consider the system of linear equations:

$$\begin{aligned} x - y - z &= 2 \\ 3x - 3y + 2z &= 16 \\ 2x - y + z &= 9. \end{aligned}$$

- The augmented matrix for this system is

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{bmatrix}.$$



## Example (cont'd)

- Multiplying the first row by $-3$ and adding to the second row gives us:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -3 & 2 & 16 \\ 2 & -1 & 1 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{bmatrix}.$$

- Multiplying the first row by $-2$ and adding to the third row gives us:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 2 & -1 & 1 & 9 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{bmatrix}.$$



## Example (cont'd)

- Interchanging the second and third rows, we have:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 5 & 10 \\ 0 & 1 & 3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix}.$$

- Dividing the third row by 5, we have:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 5 & 10 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

- Note: The arrow indicates that the corresponding systems of equations are equivalent, not that the matrices are equal.



## Example (cont'd)

- System of equations which corresponds to the final matrix:

$$\begin{aligned} x - y - z &= 2 \\ y + 3z &= 5 \\ z &= 2, \end{aligned}$$

- From this system, we obtain (by *back-substitution*)

$$\begin{aligned} z &= 2 \\ y &= 5 - 6 = -1 \\ x &= 2 - 1 + 2 = 3. \end{aligned}$$

## Example (cont'd)

- That is, our system of equations has the unique solution

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$



## Example (cont'd)

- Note: We could have continued simplifying the augmented matrix to obtain:

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

- From this we have immediately

$$\begin{aligned} x &= 3 \\ y &= -1 \\ z &= 2. \end{aligned}$$



## Terminology

- We call the procedure for solving systems of linear equations outlined above *Gaussian elimination*.
- We say a matrix is in *row-echelon* form if
  - any row of all zeros is beneath any row with nonzero entries,
  - the first nonzero entry from the left in each nonzero row is 1 (the *leading* 1 for that row),
  - each leading 1 is to the right of the all leading 1's of the rows above it.
- If in addition to being in row-echelon form, each leading 1 is the only nonzero entry in its column, we say the matrix is in *reduced row-echelon form*.
- Theorem: Every matrix may be reduced to row-echelon form (or reduced row-echelon form) by a series of elementary row operations.



## Solving linear systems

- Given a system of linear equations, use elementary row operations to reduce the augmented matrix to either row-echelon form or reduced row-echelon form.
- Assign a parameter to each variable which does not correspond to a leading 1.
- Solve for the variables which correspond to leading 1's, perhaps using back-substitution.

## Example

- To solve the system

$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 &= 3 \\ -x_1 + x_2 - x_3 &= -3, \end{aligned}$$

we first create the augmented matrix

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}.$$



## Example (cont'd)

- Applying elementary row operations, we have

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- We now have the equivalent system

$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 1 \\ x_3 - x_4 &= 1 \end{aligned}$$



## Example (cont'd)

- The *leading variables* are $x_1$ and $x_3$.
- We may assign arbitrary values to the nonleading variables, say, $x_2 = s$ and $x_4 = t$.
- Using back-substitution, we have

$$\begin{aligned} x_1 &= s - t + 2 \\ x_2 &= s \\ x_3 &= t + 1 \\ x_4 &= t. \end{aligned}$$



## Example (cont'd)

- That is, the solution is

$$X = \begin{bmatrix} s - t + 2 \\ s \\ t + 1 \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

- Geometrically, the solution is a plane in four dimensional space.

## Example (cont'd)

- Note: If we had gone to reduced row-echelon form, we would have had the matrix
$$\begin{bmatrix} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$
from which we could have read off the same solution.
- Note: If $A$ is the augmented matrix, the Octave command `rref(A)` will find the reduced row-echelon form of $A$.