Mathematics 160: Lecture 30

Full Rank

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Consequences of the rank theorem

- If A is an $m \times n$ matrix, then $\operatorname{rank} A \leq m$ and $\operatorname{rank} A \leq n$.
- An $n \times n$ matrix A is invertible if and only if $\operatorname{rank} A = n$.
- For any matrix A, $\operatorname{rank} A^T = \operatorname{rank} A$.

Theorem

- If A is an $m \times n$ matrix, U is an invertible $m \times m$ matrix, and V is an invertible $n \times n$ matrix, then $\operatorname{rank} A = \operatorname{rank} UAV$.
- Reason:
  - If $B = UA$, then $\operatorname{rank} A = \operatorname{rank} B$ since U is the product of elementary matrices.
  - Similarly, $\operatorname{rank} A^T = \operatorname{rank} V^T B^T = \operatorname{rank}(BV)^T = \operatorname{rank}(UAV)^T$.
  - Hence $\operatorname{rank} A = \operatorname{rank} UAV$.
  - Note: in particular, if A and B are similar, then $\operatorname{rank} A = \operatorname{rank} B$.

Theorem

- Let A be an $m \times n$ matrix with columns $\vec{c}_1$, $\vec{c}_2$, ..., $\vec{c}_n$. The following are equivalent:
  1. the only solution of $A\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.
  2. $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is linear independent.
  3. $\operatorname{rank} A = n$.
  4. the $n \times n$ matrix $A^T A$ is invertible.

Theorem (cont'd)

- Reason:
  - (1) $\Rightarrow$ (2): If $t_1\vec{c}_1 + t_2\vec{c}_2 + \cdots + t_n\vec{c}_n = \vec{0}$, then $A\vec{t} = \vec{0}$ where $\vec{t} = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix}^T$. Hence $\vec{t} = \vec{0}$, and $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is linearly independent.
  - (2) $\Rightarrow$ (3): rank $A = \dim(\text{col } A) = n$.

Theorem (cont'd)

- Reason (cont'd):
  - (3) $\Rightarrow$ (4): Suppose $A^TA\vec{x} = \vec{0}$ and let

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = A\vec{x}.$$

    - Then

$$0 = \vec{x}^T(A^TA\vec{x}) = (A\vec{x})^T(A\vec{x}) = \vec{y}^T\vec{y} = \vec{y}\cdot\vec{y} = \|\vec{y}\|^2.$$

    - Hence $\vec{y} = \vec{0}$, and so $\vec{x} = \vec{0}$ since $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\}$ is linearly independent.
    - Thus A^TA is invertible.
  - (4) $\Rightarrow$ (1): If $A\vec{x} = \vec{0}$, then

$$(A^TA)\vec{x} = A^T(A\vec{x}) = A^T\vec{0} = \vec{0},$$

and so $\vec{x} = \vec{0}$ since A^TA is invertible.

Theorem

- Let A be an $m \times n$ matrix with columns $\vec{c}_1$, $\vec{c}_2$, ..., $\vec{c}_n$. The following are equivalent:
  1. $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbb{R}^m$.
  2. $\text{span}\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_n\} = \mathbb{R}^m$.
  3. rank $A = m$.
  4. the $m \times m$ matrix AA^T is invertible.

Theorem (cont'd)

- Reason:
  - (1) $\Rightarrow$ (2): We have $\mathbb{R}^m = \text{im } A = \text{col } A$.
  - (2) $\Rightarrow$ (3): rank $A = \dim(\text{col } A) = m$.
  - (3) $\Rightarrow$ (4): Since rank $A^T = m$, by the previous theorem $(A^T)^TA^T = AA^T$ is invertible.
  - (4) $\Rightarrow$ (1): Given $\vec{b}$ in $\mathbb{R}^m$, there is a $\vec{y}$ in $\mathbb{R}^m$ such that $(AA^T)\vec{y} = \vec{b}$. Hence $A\vec{x} = \vec{b}$, where $\vec{x} = A^T\vec{y}$.

Theorem

- For any $m \times n$ matrix A,

$$\dim(\operatorname{im} A) + \dim(\operatorname{null} A) = n.$$

- Reason:
  - Recall: if $r = \operatorname{rank} A$, then the system $A\vec{x} = \vec{0}$ has $n - r$ basic solutions.
  - That is, $\dim(\operatorname{null} A) = n - r = n - \dim(\operatorname{im} A)$.

Example

- In a previous example, we saw that if

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 2 & 1 & 4 & 0 \\ 3 & 2 & -1 & 2 \end{bmatrix},$$

then

$$\left\{ \begin{bmatrix} 23 \\ -42 \\ -1 \\ 7 \end{bmatrix} \right\}$$

is a basis for null A and

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\}$$

is a basis for im A.

Example (cont'd)

- In particular, $\dim(\operatorname{null} A) = 1$, $\dim(\operatorname{im} A) = 3$, and $\dim(\operatorname{im} A) + \dim(\operatorname{null} A) = 4$.

Example

- If A is $1 \times n$, and is nonzero, then $\dim(\operatorname{im} A) = 1$, so $\dim(\operatorname{null} A) = n - 1$.
  - Note: if $n = 2$, $A\vec{x} = 0$ is the equation of a line through the origin.
  - Note: if $n = 3$, $A\vec{x} = 0$ is the normal equation of a plane through the origin.