Mathematics 160: Lecture 32

The Gram-Schmidt Algorithm

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Theorem

- Suppose $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$ is an orthogonal set in $\mathbb{R}^n$. If $\vec{x}$ is in $\mathbb{R}^n$ and we let

$$\vec{f}_{k+1} = \vec{x} - \left(\frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 + \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2} \vec{f}_2 + \cdots + \frac{\vec{x} \cdot \vec{f}_k}{\|\vec{f}_k\|^2} \vec{f}_k \right),$$

  then $\vec{f}_{k+1} \cdot \vec{f}_i = 0$ for all $i = 1, 2, \ldots, k$.
- Reason:

$$\begin{aligned} \vec{f}_{k+1} \cdot \vec{f}_i &= \left(\vec{x} - \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 - \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2} \vec{f}_2 - \cdots - \frac{\vec{x} \cdot \vec{f}_k}{\|\vec{f}_k\|^2} \vec{f}_k \right) \cdot \vec{f}_i \\ &= \vec{x} \cdot \vec{f}_i - \frac{\vec{x} \cdot \vec{f}_i}{\|\vec{f}_i\|^2} (\vec{f}_i \cdot \vec{f}_i) \\ &= 0. \end{aligned}$$

Consequences

- If $\vec{x}$ is in $\operatorname{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$, then

$$\vec{x} = \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 + \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2} \vec{f}_2 + \cdots + \frac{\vec{x} \cdot \vec{f}_k}{\|\vec{f}_k\|^2} \vec{f}_k$$

  and $\vec{f}_{k+1} = \vec{0}$.
- If $\vec{x}$ is not in $\operatorname{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$, then $\vec{f}_{k+1} \neq \vec{0}$ and $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k, \vec{f}_{k+1}\}$ is an orthogonal set.

Theorem

- If $B = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\}$ is an orthogonal subset of a subspace U of $\mathbb{R}^n$, then B may be enlarged to form an orthogonal basis of U.
- In particular, every proper subspace of $\mathbb{R}^n$ has an orthogonal basis.

Gram-Schmidt Algorithm

- Given a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m\}$ of a subspace U of $\mathbb{R}^n$, define

$$\begin{aligned}
\vec{f}_1 &= \vec{x}_1, \\
\vec{f}_2 &= \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1, \\
\vec{f}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2, \\
&\vdots \\
\vec{f}_m &= \vec{x}_m - \frac{\vec{x}_m \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_m \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 - \cdots - \frac{\vec{x}_m \cdot \vec{f}_{m-1}}{\|\vec{f}_{m-1}\|^2}\vec{f}_{m-1}.
\end{aligned}$$

Then
  - $\text{span}\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_k\} = \text{span}\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ for each $k = 1, 2, \ldots, m$, and
  - $\{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_m\}$ is an orthogonal basis for U.

Example

- Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ with equation $\vec{p} = s\vec{v} + t\vec{w}$, where

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}.$$

- That is, $\mathcal{P} = \text{span}\{\vec{v}, \vec{w}\}$.
- If we let

$$\vec{u} = \vec{w} - \frac{\vec{w} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} - \frac{4}{6}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{5}{3} \\ -\frac{1}{3} \\ \frac{7}{3} \end{bmatrix},$$

then $\{\vec{v}, \vec{u}\}$ is an orthogonal basis for $\mathcal{P}$.

Example (cont'd)

- Note:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -1 \\ 7 \end{bmatrix} \right\}$$

is also an orthogonal basis for $\mathcal{P}$.
- And

$$\left\{ \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{5\sqrt{3}}\begin{bmatrix} -5 \\ -1 \\ 7 \end{bmatrix} \right\}$$

is an orthonormal basis for $\mathcal{P}$.

Example

- Let U be the subspace in $\mathbb{R}^4$ with basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix} \right\}$$

- To find an orthogonal basis for U, we begin with

$$\vec{f}_1 = \vec{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

and

$$\vec{f}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} - \frac{7}{10}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -7 \\ 6 \\ 3 \\ -4 \end{bmatrix}.$$

Example (cont'd)

- Next,

$$\begin{aligned}\vec{f}_3 &= \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 \\ &= \begin{bmatrix}1\\0\\4\\2\end{bmatrix} - \frac{9}{10}\begin{bmatrix}1\\2\\1\\2\end{bmatrix} + \frac{3}{110}\begin{bmatrix}-7\\6\\3\\-4\end{bmatrix} \\ &= \frac{1}{11}\begin{bmatrix}-1\\-18\\35\\1\end{bmatrix}.\end{aligned}$$

- Then $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$ is an orthogonal basis for U.

Example (cont'd)

- Note: we also have that

$$\left\{\begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \begin{bmatrix}-7\\6\\3\\-4\end{bmatrix}, \begin{bmatrix}-1\\-18\\35\\1\end{bmatrix}\right\}$$

is an orthogonal basis for U.

- We then have that

$$\left\{\frac{1}{\sqrt{10}}\begin{bmatrix}1\\2\\1\\2\end{bmatrix}, \frac{1}{\sqrt{110}}\begin{bmatrix}-7\\6\\3\\-4\end{bmatrix}, \frac{1}{\sqrt{1551}}\begin{bmatrix}-1\\-18\\35\\1\end{bmatrix}\right\}$$

is an orthonormal basis for U.