Mathematics 160: Lecture 4

Homogeneous Systems

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Rank

- We will eventually prove that the reduced row-echelon form of a matrix is unique.
- The row echelon form of a matrix is not unique, but the number of leading 1's in a row echelon form is unique.
- Definition: We call the number of leading 1's in a row echelon form of a matrix A the *rank* of A, which we denote $\text{rank}(A)$.
- Note: Consider a consistent system of m linear equations in n variables. If r is the rank of the augmented matrix, then the system has r leading variables and $n-r$ nonleading variables. Hence the set of solutions has $n-r$ parameters.

Example

- In a previous example, we saw that the row echelon form of

$$A = \begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{bmatrix}.$$

is

$$\begin{bmatrix} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- Hence $\text{rank}(A) = 2$.

Solution sets

- Theorem: Given a system of linear equations, there are either
  - no solutions,
  - exactly one solution,
  - an infinite number of solutions.

Definitions

- We say a system of linear equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

  is *homogeneous*.
- Note: a homogeneous system always has the *trivial solution*

$$X = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Nontrivial solutions

- Theorem: Consider a homogeneous system of m linear equations and n variables. Let r be the rank of the coefficient matrix. If $r < n$, then the system has nontrivial solutions.
- Note: in particular, if $m < n$, then the system has nontrivial solutions.

Solution sets

- Suppose the rank of the augmented matrix for a homogeneous system of linear equations in n variables is r.
- Assign parameters $s_1, s_2, \ldots, s_{n-r}$ to the $n - r$ nonleading variables.
- Then every solution to the system is of the form

$$X = s_1X_1 + s_2X_2 + \cdots + s_{n-r}X_{n-r},$$

  where $X_1, X_2, \ldots, X_{n-r}$ are $n \times 1$ column matrices.
- Note: each of $X_1, X_2, \ldots, X_{n-r}$ is itself a solution of the system. We call these the *basic solutions*.
- In short: every solution to a homogeneous system of linear equations is a linear combination of basic solutions.

Example

- The homogeneous system

$$\begin{aligned} x_1 + 2x_2 - 2x_3 - 12x_4 &= 0 \\ x_1 + 3x_2 - 18x_4 + 6x_5 &= 0 \\ x_1 - 6x_3 - 12x_5 &= 0 \end{aligned}$$

  has augmented matrix

$$\begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 0 \\ 1 & 3 & 0 & -18 & 6 & 0 \\ 1 & 0 & -6 & 0 & -12 & 0 \end{bmatrix}.$$

Example (cont'd)

- Applying row operations, the augmented matrix becomes

$$\begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 0 \\ 0 & 1 & 2 & -6 & 6 & 0 \\ 0 & -2 & -4 & 12 & -12 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -2 & -12 & 0 & 0 \\ 0 & 1 & 2 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

- If we let $x_3 = s$, $x_4 = t$, and $x_5 = u$, then $x_2 = -2s + 6t - 6u$ and $x_1 = -2(-2s + 6t - 6u) + 2s + 12t = 6s + 12u$.

Example (cont'd)

- Hence the solutions are

$$X = s\begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u\begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- The basic solutions are

$$X_1 = \begin{bmatrix} 6 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, X_2 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}, X_3 = \begin{bmatrix} 12 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Example

- Recall: given real numbers a, b, c, d, e, and f, with not all of a, b, and c zero, the graph of the equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

is a conic section.

- To find a conic section passing through the five points (p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4), and (p_5, q_5), we would need to solve

$$\begin{aligned} p_1^2 a + p_1 q_1 b + q_1^2 c + p_1 d + q_1 e + f &= 0 \\ p_2^2 a + p_2 q_2 b + q_2^2 c + p_2 d + q_2 e + f &= 0 \\ p_3^2 a + p_3 q_3 b + q_3^2 c + p_3 d + q_3 e + f &= 0 \\ p_4^2 a + p_4 q_4 b + q_4^2 c + p_4 d + q_4 e + f &= 0 \\ p_5^2 a + p_5 q_5 b + q_5^2 c + p_5 d + q_5 e + f &= 0. \end{aligned}$$

Example (cont'd)

- Since this is a linear system of 5 equations in 6 variables, by the previous theorem, there exist nontrivial solutions. If the five points do not lie on a line, then we cannot have a, b, and c all zero.
- Conclusion: given five points in the plane, not all on a line, there exists a conic passing through them.

Example

- To find a conic section passing through the points $(2,1)$, $(-2,1)$, $(1,3)$, $(1,-3)$, and $(-1,3)$, we need to solve

$$\begin{aligned} 4a+2b+c+2d+e+f &= 0\\ 4a-2b+c-2d+e+f &= 0\\ a+3b+9c+d+3e+f &= 0\\ a-3b+9c+d-3e+f &= 0\\ a-3b+9c-d+3e+f &= 0. \end{aligned}$$

Example (cont'd)

- Using Octave, we row reduce the augmented matrix:

$$\begin{bmatrix} 4 & 2 & 1 & 2 & 1 & 1 & 0\\ 4 & -1 & 1 & -2 & 1 & 1 & 0\\ 1 & 3 & 9 & 1 & 3 & 1 & 0\\ 1 & -3 & 9 & 1 & -3 & 1 & 0\\ 1 & -3 & 9 & -1 & 3 & 1 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1.0000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.22857 & 0.00000\\ 0.0000 & 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000\\ 0.0000 & 0.00000 & 1.00000 & 0.00000 & 0.00000 & 0.08571 & 0.00000\\ 0.0000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 & 0.00000 & 0.00000\\ 0.0000 & 0.00000 & 0.00000 & 0.00000 & 1.00000 & 0.00000 & 0.00000 \end{bmatrix}.$$

Example (cont'd)

- Hence, for any real number s we have solutions

$$X = s\begin{bmatrix} -0.22857\\ 0\\ -0.08571\\ 0\\ 0\\ 1 \end{bmatrix}$$

- For example, with $s=-1$, we have the equation

$$0.22857x^2 + 0.08571y^2 = 1,$$

an ellipse.

Example (cont'd)

- Graph of $0.22857x^2 + 0.08571y^2 = 1$: