Mathematics 160: Lecture 5

Matrix Multiplication

Dan Sloughter

Furman University

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Example

- Two companies, A and B, each purchase three different types of widgets, α, β, and γ, to manufacture their respective products. They may purchase their widgets from one of two suppliers, I and II.

Example (cont'd)

- The following two arrays specify the demand that each company has for each type of widget and the price each supplier charges for the respective types of widgets:

	Widget α	Widget β	Widget γ
Company A	600	300	1000
Compnay B	400	800	500

	Supplier I	Supplier II
Widget α	1.00	1.50
Widget β	4.00	3.00
Widget γ	1.00	2.00

Example (cont'd)

- The cost for Company A to purchase widgets from Supplier I is then

$$600 \times 1.00 + 300 \times 4.00 + 1000 \times 1.00 = 2800.00$$

and the cost from Supplier II is

$$600 \times 1.50 + 300 \times 3.00 + 1000 \times 2.00 = 3800.00.$$

- For Company B, the cost from Supplier I is

$$400 \times 1.00 + 800 \times 4.00 + 500 \times 1.00 = 4100.00$$

and from Supplier II is

$$400 \times 1.50 + 800 \times 3.00 + 500 \times 2.00 = 4000.00.$$

Example (cont'd)

- Note: we may also present the preceding costs in an array:

	Supplier I	Supplier II
Company A	2800.00	3800.00
Company B	4100.00	4000.00

Example (cont'd)

- Now let D be the demand matrix formed from our first array, P be the price matrix formed from our second array, and C be the cost matrix formed from the last array.
- That is, let

$$D = \begin{bmatrix} 600 & 300 & 1000 \\ 400 & 800 & 500 \end{bmatrix}, P = \begin{bmatrix} 1.00 & 1.50 \\ 4.00 & 3.00 \\ 1.00 & 2.00 \end{bmatrix},$$

$$\text{and } C = \begin{bmatrix} 2800.00 & 3800.00 \\ 4100.00 & 4000.00 \end{bmatrix}.$$

Example (cont'd)

- If we let $D = [d_{ij}]$, $P = [p_{ij}]$, and $C = [c_{ij}]$, then we have

$$\begin{aligned} c_{11} &= d_{11}p_{11} + d_{12}p_{21} + d_{13}p_{31}, \\ c_{21} &= d_{21}p_{11} + d_{22}p_{21} + d_{23}p_{31}, \\ c_{12} &= d_{11}p_{12} + d_{12}p_{22} + d_{13}p_{32}, \\ c_{22} &= d_{21}p_{12} + d_{22}p_{22} + d_{23}p_{32}. \end{aligned}$$

- We will call C the product of D and P and write $C = DP$.

Dot product

- If $R = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}$ is a $1 \times n$ row matrix and

$$C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

is an $n \times 1$ column matrix, then their *dot product* is the scalar

$$RC = r_1c_1 + r_2c_2 + \cdots + r_nc_n.$$

Matrix product

- If $A = [a_{ij}]$ is an $m \times n$ matrix and $B = [b_{ij}]$ is an $n \times p$ matrix, then the *product* of A and B is the $m \times p$ matrix whose (i, j)-entry is the dot product of the ith row of A with the jth column of B.
- That is, if $C = [c_{ij}] = AB$, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Examples

- We have

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ -1 & 10 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 38 \\ -5 & -22 \end{bmatrix}.$$

- And

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 6 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 7 \\ 18 & -2 \end{bmatrix},$$

  but

$$\begin{bmatrix} 0 & 5 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ 5 & 15 \end{bmatrix}.$$

- Note: AB may not be equal to BA even if both exist and are the same size.

Definition

- We call the $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

  the *identity matrix*.
- If A is $m \times n$, then $AI_n = A$.
- If B is $n \times p$, then $I_nB = B$.

Properties

- Let c be a scalar and let A, B, and C be matrices. Assuming all indicated products are defined, we have
  - $A(BC) = (AB)C$,
  - $A(B + C) = AB + AC$,
  - $(B + C)A = BA + CA$,
  - $c(AB) = (cA)B = A(cB)$,
  - $(AB)^T = B^TA^T$.

Octave

- In Octave, if `A` is an $m \times n$ matrix and `B` is an $n \times p$ matrix, then `A*B` will compute the matrix product A and B.
- If `A` is a square matrix, then, for example, `A^3` will compute the product of `A` with itself 3 times.

System of linear equations

- We may now write the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Systems of linear equations (cont'd)

- More simply, if

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then we may write the system of equations as $AX = B$.