Mathematics 160: Lecture 6

Matrix Products: Applications

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Particular solutions

- Let A be an $m \times n$ matrix and B be an $m \times 1$ column matrix.
- Suppose X_0 is a solution of $AX = B$ and X' is a solution of $AX = O$, where O is the $m \times 1$ matrix of zeros.
- Then
$$A(X' + X_0) = AX' + AX_0 = O + B = B,$$
so $X' + X_0$ is also a solution of $AX = B$.
- Now suppose X'' is any solution of $AX = B$. Then
$$A(X'' - X_0) = AX'' - AX_0 = B - B = O.$$
- Hence $X'' - X_0$ is a solution of the homogeneous equation $AX = O$.
- Hence $X'' = X' + X_0$, where X' is some solution to $AX = 0$.

Theorem

- Let X_0 be a fixed solution to the system $AX = B$. Then every solution of $AX = B$ is of the form $X = X' + X_0$, where X' is a solution of the homogeneous system $AX = O$.
- That is, the general solution of the nonhomogenous system is the general solution of the homogeneous solution plus any particular solution of the nonhomogeneous equation.
- Explicitly, if $X_1, X_2, \ldots, X_k$ are the basic solutions of the homogeneous equation and X_0 is a particular solution of the nonhomogeneous equation, then the general solution of the nonhomogeneous system is
$$X = s_1 X_1 + s_2 X_2 + \cdots s_k X_k + X_0.$$

Example

- We saw in a previous example that the general solution of the system
$$\begin{aligned} x_1 - x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 - 2x_2 - x_3 + 3x_4 &= 3 \\ -x_1 + x_2 - x_3 &= -3, \end{aligned}$$
is
$$X = s \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Example (cont'd)

- The column matrices

$$X_1 = \begin{bmatrix}1\\1\\0\\0\end{bmatrix} \text{ and } X_2 = \begin{bmatrix}-1\\0\\1\\1\end{bmatrix}$$

are the basic solutions to the homogeneous equation and the column matrix

$$X_0 = \begin{bmatrix}2\\0\\1\\0\end{bmatrix}$$

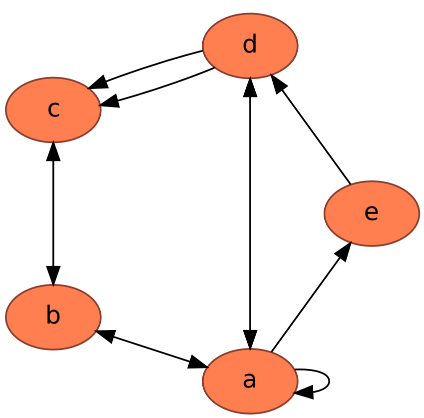
is a particular solution to the nonhomogeneous equation.

Directed graphs

- Definition: A *directed graph* is set of points, which we call *vertices*, connected by directed line segments, which we call *edges*.
- Note: directed graphs are mathematical abstractions of network systems, such as an airline route system, a city bus route system, or a telephone network.

Example

- Here is a drawing of a simple directed graph:



Example (cont'd)

- We may encode the information contained in the graph in the *adjacency matrix* A of the graph as follows: if the graph has n vertices, then A is an $n \times n$ matrix where the (i, j)-th entry is the number of edges which start at the ith vertex and to go to the jth vertex.
- For our example, the adjacency matrix is

$$A = \begin{bmatrix}1 & 1 & 0 & 1 & 1\\1 & 0 & 1 & 0 & 0\\0 & 1 & 0 & 0 & 0\\1 & 0 & 2 & 0 & 0\\0 & 0 & 0 & 1 & 0\end{bmatrix}.$$

Example (cont'd)

- Suppose we want to know the number of ways to get from vertex d to vertex b in 2 steps.
- Note: this will be sum of the number of ways to get from d to any other vertex v in one step times the number of ways to get from v to d in one step. In this case, $(1)(1) + (2)(1) = 3$.
- That is, the number of ways to get from d to b in 2 steps is the dot product of the row of A corresponding to d and the column of A corresponding to b.
- Hence if $A^2 = [b_{ij}]$, then

$$b_{ij} = \text{number of ways to get from the } i\text{th vertex to the } j\text{th vertex in 2 steps.}$$

Example (cont'd)

- Hence we can read off the 2 step adjacencies from

$$A^2 = \begin{bmatrix} 3 & 1 & 3 & 2 & 1 \\ 1 & 2 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 & 0 \end{bmatrix}.$$

Example (cont'd)

- More generally, A^n gives the number of ways to get between the vertices in n steps.
- For our example,

$$A^{10} = \begin{bmatrix} 2755 & 2118 & 2293 & 1649 & 1158 \\ 1406 & 1101 & 1163 & 847 & 599 \\ 599 & 451 & 502 & 356 & 248 \\ 1654 & 1301 & 1366 & 998 & 707 \\ 707 & 528 & 594 & 419 & 291 \end{bmatrix}$$

- For example, there are 1163 ways to get from b to c in 10 steps.