Mathematics 160: Lecture 7

Matrix Inverses

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Definition

- We say an $n \times n$ matrix A is *invertible* if there exists a matrix C for which $CA = I_n$ and $AC = I_n$.
- Note: Suppose A is invertible and C and D are matrices for which $CA = I_n = AC$ and $DA = I_n = AD$. Then

$$D = DI_n = D(AC) = (DA)C = I_nC = C.$$

- In other words, the matrix C in the definition is unique. We call it the *inverse* of A , which we denote A^{-1}.

Example

- If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then, provided $ad - bc \neq 0$,

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

which may be verified easily by multiplication.

- For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{4-6} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -4 & 2 \\ 3 & -1 \end{bmatrix}.$$

Example

- Note: not every matrix has an inverse.
- Example: the $n \times n$ matrix with all entries 0 does not have an inverse.
- There are other matrices without inverses. For example, for any real numbers a, b, c, and d,

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ 0 & 0 \end{bmatrix},$$

and so

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

cannot have an inverse.

Inverses and systems

- Note: If A is invertible, we may solve a system $AX = B$ by multiplying on the left by A^{-1}:
  - $A^{-1}B$ is a solution since
    $$A(A^{-1}B) = (AA^{-1})B = IB = B.$$
  - And if X is a solution of $AX = B$, then
    $$X = IX = \left(A^{-1}A\right)X = A^{-1}(AX) = A^{-1}B.$$
  - Hence $A^{-1}B$ is the unique solution of $AX = B$.

Example

- To solve
  $$\begin{aligned} 3x + 4y &= 6 \\ x - y &= 4, \end{aligned}$$
  let
  $$A = \begin{bmatrix} 3 & 4 \\ 1 & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$
- Then
  $$A^{-1} = -\frac{1}{7}\begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix},$$
  and so the solution is
  $$X = -\frac{1}{7}\begin{bmatrix} -1 & -4 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \end{bmatrix} = -\frac{1}{7}\begin{bmatrix} -22 \\ 6 \end{bmatrix}.$$

Some properties

- Note: Assume all matrices below are $n \times n$.
- I_n is invertible, with $I_n^{-1} = I_n$.
- If A is invertible, then A^{-1} is invertible, with $\left(A^{-1}\right)^{-1} = A$.
  - Reason: $AA^{-1} = I_n = A^{-1}A$.
- If A and B are invertible, then AB is invertible, with $(AB)^{-1} = B^{-1}A^{-1}$.
  - Reason:
    $$AB\left(B^{-1}A^{-1}\right) = A\left(BB^{-1}\right)A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$
    and
    $$B^{-1}A^{-1}(AB) = B^{-1}\left(A^{-1}A\right)B^{-1} = B^{-1}I_nB = B^{-1}B = I_n.$$
- If $A_1, A_2, \ldots, A_k$ are invertible, then $A_1A_2\cdots A_k$ is invertible, with
  $$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

Some properties (cont'd)

- If A is invertible, then, for any nonnegative integer k, A^k is invertible, with $\left(A^k\right)^{-1} = (A^{-1})^k$.
- If A is invertible, then A^T is invertible, with $\left(A^T\right)^{-1} = (A^{-1})^T$.
  - Reason:
    $$A^T\left(A^{-1}\right)^T = \left(A^{-1}A\right)^T = I_n^T = I_n$$
    and
    $$\left(A^{-1}\right)^T A^T = \left(AA^{-1}\right)^T = I_n^T = I_n.$$
- If A is invertible and $c \neq 0$ is a scalar, then cA is invertible, with
  $$(cA)^{-1} = \frac{1}{c}A^{-1}.$$

Theorem

- In the following theorems, assume A is an $n \times n$ matrix and O is a matrix, of the appropriate size, of all zeros.
- If A is invertible, then the only solution to $AX = O$ is the trivial solution $X = 0$.
- Reason: From above, the unique solution is $A^{-1}O = O$.

Theorem

- If the only solution to $AX = O$ is $X = O$, then the reduced row-echelon form of A is I_n.
- Reason: We must have $\text{rank}(A) = n$, since otherwise $AX = O$ would have non-trivial solutions. Hence the reduced row-echelon form of A must be I.

Theorem

- If the reduced row-echelon form of A is I_n, then $AX = B$ has a solution for every $n \times 1$ column matrix B.
- Reason: Reducing the augmented matrix to reduced row-echelon form produces a solution (in fact, a unique solution) in all cases.

Theorem

- If $AX = B$ has a solution for every $n \times 1$ column matrix B, then there exists an $n \times n$ matrix C for which $AC = I_n$.
- Reason:
  - For $j = 1, 2, \ldots, n$, let E_j be the $n \times 1$ column matrix with entries

$$e_{i1} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

  - Let C_j be a $n \times 1$ column which is a solution to $AX = E_j$ for $j = 1, 2, \ldots, n$.
  - Let C be the matrix with columns C_1, C_2, ..., C_n.
  - Then $AC = I_n$.

Theorem

- If there exists an $n \times n$ matrix C for which $AC = I_n$, then A is invertible.
- Reason:
  - First note that if $CX = O$, then
$$X = I_n X = (AC)X = I_n O = O.$$
  - Hence the only solution of $CX = O$ is $X = O$.
  - It follows from the previous theorems that there exists an $n \times n$ matrix D such that $CD = I_n$.
  - But then
$$A = AI_n = A(CD) = (AC)D = I_n D = D.$$
  - Hence $CA = I_n$.
  - Since we have assumed $AC = I_n$, it follows that A is invertible with $A^{-1} = C$.

Summary of previous results

- Suppose A is an $n \times n$ matrix and O is a matrix, of the appropriate size, of all zeros.
- The following statements are equivalent:
  - A is invertible.
  - The only solution to $AX = O$ is the trivial solution $X = 0$.
  - The reduced row-echelon form of A is I_n.
  - $AX = B$ has a solution for every $n \times 1$ column matrix B.
  - There exists a $n \times n$ matrix C for which $AC = I_n$.