Mathematics 160: Lecture 8

Inverses and Elementary Matrices

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Computing an inverse

- The proofs of the previous lecture provide a method for finding inverses.
- Namely: given an $n \times n$ matrix A, let C_1, C_2, ..., C_n be the solutions of $AX = E_j$, where, for $j = 1, 2, \ldots, n$, E_j is the $n \times 1$ column matrix $[e_{i1}]$ with
$$e_{i1} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$
Then A^{-1} is the matrix with columns C_1, C_2, ..., C_n.
- Note: this requires solving n systems of n equations in n unknowns.
- Note: we can do this at once by writing an augmented matrix consisting of A and I_n and reducing A to I_n.
- That is, the same row operations which take A to I_n take I_n to A^{-1}.

Example

- We will find the inverse of
$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}.$$
- Using elementary row operations, we find
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

Example (cont'd)

- Continuing, we have
$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$
$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 5 & -1 & 1 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right].$$
- Hence
$$A^{-1} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}.$$

Example

- Suppose we wish to find an inverse for

$$A = \begin{bmatrix} 2 & 1 & -4 \\ -4 & -1 & 6 \\ -2 & 2 & -2 \end{bmatrix}.$$

- Using elementary row operations, we find that

$$\left[\begin{array}{rrrrrr} 2 & 1 & -4 & 1 & 0 & 0 \\ -4 & -1 & 6 & 0 & 1 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrrr} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right]$$

Example (cont'd)

- Continuing, we have

$$\left[\begin{array}{rrrrrr} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 3 & -6 & 1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{rrrrrr} 2 & 1 & -4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & -5 & -3 & 1 \end{array}\right].$$

- Note: this shows that it is not possible to reduce A to I_3 using elementary row operations.
- It follows that A does not have an inverse.

Inverses in Octave

- To find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix},$$

we can either
  - apply `rref` to the augmented matrix

$$\left[\begin{array}{rrrrrr} 1 & 2 & 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 1 & 0 \\ 4 & 2 & 1 & 0 & 0 & 1 \end{array}\right],$$

  - or use `inv(A)`.
  - In either case, we find that

$$A^{-1} = \begin{bmatrix} -1.0000 & 2.0000 & 1.0000 \\ 2.5000 & -5.5000 & -2.0000 \\ -1.0000 & 3.0000 & 1.0000 \end{bmatrix}$$

Definition

- An *elementary matrix* is any matrix which may be obtained by applying an elementary row operation to an identity matrix.
- Examples: Each of the following is an elementary matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example

- Let
$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{bmatrix} \text{ and } E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$
- Note: E is an elementary matrix obtained by multiplying the second row of I_3 by 3.
- Note:
$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 12 & 15 & 18 & 6 \\ 7 & 8 & 9 & 1 \end{bmatrix},$$
which is what we would obtain applying the same elementary row operation to A.

Example (cont'd)

- Let
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
the elementary 3×3 matrix obtained by adding 3 times the first row to the second row of I_3.
- Now
$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 7 & 11 & 15 & 5 \\ 7 & 8 & 9 & 1 \end{bmatrix},$$
which is what we would obtain from applying the same row operation to A.

Example (cont'd)

- Let
$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$
the elementary 3×3 matrix obtained by interchanging the second and third rows of I_3.
- Then
$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 7 & 8 & 9 & 1 \\ 4 & 5 & 6 & 2 \end{bmatrix},$$
which is what we would obtain from applying the same row operation to A.

Row operations

- Suppose an elementary row operation is performed on an $m \times n$ matrix A to obtain the matrix B.
- Let E be the elementary matrix obtained by performing the same row operation on I_m.
- Then $B = EA$.

Inverses of elementary matrices

- Every elementary matrix E is invertible, and E^{-1} is itself an elementary matrix:
  - If E multiplies a row by a nonzero scalar c, E^{-1} multiplies that same row by $\frac{1}{c}$.
  - If E multiplies row i by scalar c and adds it to row j, E^{-1} multiplies row i by $-c$ and adds it to row j.
  - If E interchanges rows i and j, E^{-1} interchanges rows i and j (and, so, $E = E^{-1}$).

Theorem

- Suppose an $m \times n$ matrix A is reducible to an $m \times n$ matrix B by a series of elementary row operations. Then:
  - $B = E_k E_{k-1} \cdots E_1 A$, where E_1, E_2, ..., E_k are elementary matrices.
  - $B = UA$ for some invertible matrix U.
  - U is the matrix obtained from I_m by performing the row operations on I_m which reduce A to B.

Theorem

- If A is an invertible matrix, then there exist elementary matrices E_1, E_2, ... E_k such that
$$A = E_1 E_2 \cdots E_k.$$
- Reason:
  - Since A is reducible to I, there exist elementary matrices D_1, D_2, ..., D_k such that
$$I = D_k D_{k-1} \cdots A.$$
  - Hence
$$A = D_1^{-1} D_2^{-1} \cdots D_k^{-1}.$$
  - So let $E_i = D_i^{-1}$, $i = 1, 2, \ldots, k$.

Theorem

- If R and S are both reduced row-echelon forms of a matrix A, then $R = S$.
- We will not prove this, but the proof is based on the following observations:
  - There exist invertible matrices P and Q such that $R = PA$ and $S = QA$.
  - Hence $A = Q^{-1}S$ and $A = P^{-1}R$, so $S = QP^{-1}R$.
  - That is, $S = UR$ for some invertible matrix U.