Mathematics 160: Lecture 9

Markov Chains

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Example

- Consider a bit, 0 or 1, which is to be transmitted through a series of nodes. At each node, there is a 10% chance that the an error will be made in relaying the bit.
- That is, a 0 is changed into a 1 or a 1 into 0 10% of the time.
- We may encode this information in a matrix: let $P = [p_{ij}]$ where

$$p_{ij} = \text{ probability that } j \text{ becomes } i.$$

- In this case,

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

Example (cont'd)

- Let $P^{(2)} = [p_{ij}^{(2)}]$, where

$$p_{ij}^{(2)} = \text{probability that } j \text{ becomes } i \text{ after 2 steps.}$$

- These are the *two-step* probabilities.
- Then

$$\begin{aligned}
p_{00}^{(2)} &= p_{00}p_{00} + p_{01}p_{10} = (0.9)(0.9) + (0.1)(0.1) = 0.82, \\
p_{10}^{(2)} &= p_{10}p_{00} + p_{11}p_{10} = (0.1)(0.9) + (0.9)(0.1) = 0.18, \\
p_{01}^{(2)} &= p_{00}p_{01} + p_{01}p_{11} = (0.9)(0.1) + (0.1)(0.9) = 0.18, \\
p_{11}^{(2)} &= p_{10}p_{01} + p_{11}p_{11} = (0.1)(0.1) + (0.9)(0.9) = 0.82.
\end{aligned}$$

- That is, $P^{(2)} = P^2$.

Example (cont'd)

- In general, P^n is the matrix of *n-step* probabilities.
- For example, we have

$$P^{10} = \begin{bmatrix} 0.55369 & 0.44631 \\ 0.44631 & 0.55369 \end{bmatrix},$$

$$P^{50} = \begin{bmatrix} 0.50001 & 0.49999 \\ 0.49999 & 0.50001 \end{bmatrix},$$

$$P^{100} = \begin{bmatrix} 0.50000 & 0.50000 \\ 0.50000 & 0.50000 \end{bmatrix}.$$

- That is, after 100 steps, the state is equally likely to be 0 or 1 independent of that starting state.

Example (cont'd)

- Now suppose the initial bit is 0 with probability 0.2 and 1 with probability 0.8.
- Then

$$\text{probability it is a 0 after 1 step} = (0.2)(0.9) + (0.8)(0.1) = 0.26$$

and

$$\text{probability it is a 1 after 1 step} = (0.2)(0.1) + (0.8)(0.9) = 0.74.$$

- Note: we could compute this as

$$\begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.26 \\ 0.74 \end{bmatrix}.$$

Example (cont'd)

- That is, if S_0 is the 2×1 column matrix representing the initial probabilities and S_1 is the 2×1 column matrix representing the probabilities of the states after 1 step, then

$$S_1 = PS_0.$$

- More generally, if S_n is the column matrix giving the probabilities of the states after n steps, then

$$S_n = PS_{n-1} = P^2 S_{n-2} = \cdots = P^n S_0.$$

- Suppose

$$\lim_{n\to\infty} P^n = Q.$$

- Then

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} P^n S_0 = QS_0.$$

Example (cont'd)

- Hence $\lim_{n\to\infty} S_n$ exists. Let $S = \lim_{n\to\infty} S_n$.
- Since $S_n = PS_{n-1}$, it follows that

$$S = \lim_{n\to\infty} S_n = P \lim_{n\to\infty} S_{n-1} = PS.$$

- Note: $S = PS$ if and only if $(I - P)S = O$.
- For our example,

$$S = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

Definition

- A *finite Markov chain* is a probability model consisting of a set of *states* $\{1, 2, 3, \ldots, n\}$ and an $n \times n$ *transition matrix* $P = [p_{ij}]$, where p_{ij} specifies the probability of moving from state j to state i in one step.
- Note: the columns of P must sum to 1. That is, for $j = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n} p_{ij} = 1.$$

We say P is a *stochastic* matrix.

- Note: if $P^m = [p_{ij}^{(m)}]$, then

$$p_{ij}^{(m)} = \text{probability of being in state } i \text{ after } m \text{ steps, having started in state } j.$$

More terminology

- We call a column matrix whose entries are all nonnegative and sum to 1 a *probability vector*
- We call the probability vector

$$S_m = \begin{bmatrix} s_1^{(m)} \\ s_2^{(m)} \\ \vdots \\ s_n^{(m)} \end{bmatrix},$$

where $s_i^{(m)}$ is the probability the chain is in state i after m steps, a *state vector*.

- If, given any initial state vector S_0, we have $S = \lim_{m\to\infty} S_m$, we call S a *steady-state vector*.

Fixed vectors

- We call a state vector S such that $PS = S$ a *fixed vector* for the transition matrix P.
- Example: The state vector

$$S = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

is a fixed vector for the transition matrix

$$P = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 0.9 \end{bmatrix}.$$

- Theorem: Given a transition matrix P, there exists a fixed vector S.
- Note: $\text{rank}(I - P) < n$, so there are non-trivial solutions to $(I - P)S = O$.

Definition

- We say a Markov chain with transition matrix P is *regular* if for some m all of the entries of P^m are nonzero.
- Example: if

$$P = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0 \end{bmatrix},$$

then the Markov chain is regular since

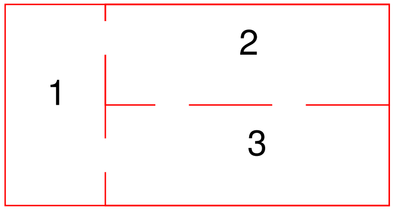
$$P^2 = \begin{bmatrix} 0.75 & 0.50 \\ 0.25 & 0.50 \end{bmatrix}.$$

Steady-state vectors

- Note: if S is a steady-state vector, then S is a fixed vector.
- Theorem: Suppose P is the transition matrix for a regular Markov chain and let S be the fixed vector for P. Then
  - S is the steady-state vector for P.
  - $\lim_{n\to\infty} P^n = Q$, where Q is an $n \times n$ matrix with all columns equal to S.

Example

- A mouse is placed in the following maze:



- When a bell is rung, the mouse randomly chooses a door and moves through it.

Example (cont'd)

- The transition matrix for the Markov chain is

$$P = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{2}{3} & 0 \end{bmatrix}.$$

- Then

$$P^2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{9} & \frac{2}{9} \\ \frac{1}{3} & \frac{11}{18} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & \frac{11}{18} \end{bmatrix},$$

so, in particular, P is a regular transition matrix.

Example (cont'd)

- To find the fixed vector, we need to solve $(I - P)S = O$.
- Using row reduction, we find

$$\begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{2} & 1 & -\frac{2}{3} & 0 \\ -\frac{1}{2} & -\frac{2}{3} & 1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{5}{6} & -\frac{5}{6} & 0 \\ 0 & -\frac{5}{6} & \frac{5}{6} & 0 \end{bmatrix}$$
$$\longrightarrow \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example (cont'd)

- Hence if we let $s_3 = t$, then $s_2 = t$ and $s_1 = \frac{2}{3}t$.
- Now we must have

$$1 = s_1 + s_2 + s_3 = \frac{2}{3}t + t + t = \frac{8}{3}t,$$

so $t = \frac{3}{8}$.

- Hence the fixed, and steady-state, vector is

$$S = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{3}{8} \end{bmatrix}.$$

Example (cont'd)

- Note: this means that

$$\lim_{n\to\infty} P^n = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{3}{8} & \frac{3}{8} \end{bmatrix}.$$

- In particular, in the long run, the mouse spends $\frac{1}{4}$ of its time in compartment 1, $\frac{3}{8}$ of its time in compartment 2, and $\frac{3}{8}$ of its time in compartment 3.